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Self-gravitational instability of a rotating Hall plasma

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The effect of rotation on the development of the hydromagnetic stability of a self-gravitating, inviscid, incompressible and infinitely conducting plasma of variable density in the presence of Hall-currents has been investigated. The solution is shown to be characterized by a variational principle. Based on the existence of this, the solution has been derived for the case of a fluid having exponentially varying density along the direction of the uniform vertical magnetic field. The dispersion relation has been solved numerically and it has been found that growth rate increases with both Hall currents and rotation, showing thereby destabilizing character of Hall currents as well as of rotation.

1. Introduction

During the last few years several authors (Talwar & Kalra 1967, Hosking 1968, Singh & Tandon 1969, Ariel 1970a, b, Bhowmik 1972) have demonstrated the destabilizing influence of Hall currents on the Rayleigh-Taylor instability problems. Ariel (1974) has studied the combined influence of Hall currents and coriolis forces on Rayleigh-Taylor instability of a plasma in which density is stratified in a direction antiparallel to that of gravity. Bhatia (1974) has examined the influence of Hall effects on the stability of a self-gravitating incompressible stratified plasma. It would, therefore, be of interest to examine the influence of Hall currents on the dynamic stability of a self-gravitating rotating plasma of variable density. This aspect forms the subject matter of this note.

2. Perturbation Equations

The relevant linearized perturbation equations are :

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} = -\Delta\delta p+\frac{1}{4\pi}[(\nabla\times \boldsymbol{h})\times \boldsymbol{H}]+\delta\rho\nabla\phi_0+\rho_0\nabla\delta\phi_0+2\rho_0(\boldsymbol{v}\times\boldsymbol{\Omega}), \qquad \ldots (1)$$

$$\frac{\partial \boldsymbol{h}}{\partial t} = \nabla\times(\boldsymbol{v}\times \boldsymbol{H})-\frac{c}{4\pi Ne}\{\nabla\times[(\nabla\times \boldsymbol{h})\times \boldsymbol{H}]\}, \qquad \ldots (2)$$

$$\frac{\partial}{\partial t}(\delta\rho)+(\boldsymbol{v}.\nabla)\rho_0 = 0, \qquad \ldots (3)$$

$$\nabla^2 \delta\phi = -4\pi G \delta\rho, \qquad \ldots (4)$$

$$\nabla \cdot \boldsymbol{v} = 0, \quad \nabla \cdot \boldsymbol{h} = 0, \qquad \ldots (5)$$

where $\boldsymbol{v}(u, v, w)$, $\boldsymbol{h}(h_x, h_y, h_z)$, $\delta\rho$, δp and $\delta\phi$ are the perturbations, respectively, in velocity, magnetic field $\boldsymbol{H}$, density ρ_0, pressure p and the gravitational potential ϕ_0. Here e and N are respectively the charge and the number density of the particles of the medium, G is the gravitational constant and $\boldsymbol{\Omega}$ is the uniform angular velocity.

Assuming that $\boldsymbol{H} = (0, 0, H_0)$ and $\boldsymbol{\Omega} = (0, 0, \Omega)$, analysing in terms of normal modes by seeking solutions of the above equations of the form

$$F(z) \exp(ik_x x + ik_y y + nt), \qquad \ldots (6)$$

where $F(z)$ is some function of z, k_x and k_y are the horizontal wave numbers of the harmonic disturbance and n is frequency, eliminating some of the variables we finally get

$$\left\{ n\rho_0 + \frac{(D\rho_0)(D\phi_0)}{n} \right\} w - \frac{n}{k^2} D(\rho_0 Dw) + \delta\phi D\rho_0$$

$$+ \frac{H_0}{4\pi k^2} (D^2 - k^2) Dh_z - \frac{2}{k^2} \Omega D(\rho_0 \zeta) = 0, \qquad \ldots (7)$$

$$nh_z = H_0 Dw - \frac{cH_0}{4\pi Ne} D\xi, \qquad \ldots (8)$$

$$n\rho_0 \zeta = \frac{H_0}{4\pi} D\xi + 2\rho_0 \Omega Dw, \qquad \ldots (9)$$

$$n\xi = H_0 D\zeta + \frac{cH_0}{4\pi Ne} (D^2 - k^2) Dh_z. \qquad \ldots (10)$$

where ζ and ξ are respectively the vertical components of the vectors curl $\boldsymbol{v}$ and curl $\boldsymbol{h}$.

Let us assume that the plasma is contained between two free boundaries. The boundary conditions that must be satisfied at the surfaces $z = 0$ and $z = d$ are (Bhatia 1974, Ariel 1974) :

$$\left. \begin{array}{l} w = 0, \quad D^2 w = 0, \\ \xi = D(h_z) = D(\zeta) = 0, \\ (D - k)\delta\phi = 0, \quad (D + k)\delta\phi = 0, \\ \dfrac{cH_0}{4\pi Ne} (D^2 - k^2) h_z + H_0 \zeta = 0 \end{array} \right\} \qquad \ldots (11)$$

A variational principle

Multiplying i-th component of eq. (7) by w_j and integrating over the vertical extent L of the plasma, and performing the integration by parts and using boundary conditions, we get

$$n_i\left[\int_L \rho_0(k^2w_iw_j+Dw_iDw_j)dz+\frac{1}{4\pi}\int_L \xi_i\xi_jdz+\frac{k^2}{n_i^2}\int_L D\phi_0D\rho_0w_iw_jdz\right]$$

$$+n_j\left[\int_L \rho_0\zeta_i\zeta_jdz+\frac{1}{4\pi}\int_L (Dh_iDh_j+k^2h_ih_j)dz\right.$$

$$\left.-\frac{k^2}{4\pi G}\int_L (D\delta\phi_iD\delta\phi_j+k^2\delta\phi_i\delta\phi_j)dz\right]=0. \quad \ldots \quad (12)$$

Setting $i=j$ in eq. (12) and considering the arbitrary variations δw, δh etc. we can show, by proceeding along the usual lines, that $\delta n=0$.

Dispersion relation

Now we shall treat the problem of an interstellar plasma in which the undisturbed density distribution is given by

$$\rho_0(z)=\rho_1\exp(\beta z), \quad \ldots \quad (13)$$

where ρ_1 and β are constants.

Poisson's equation which must be satisfied by ϕ_0, then gives the following distribution for ϕ_0.

$$\phi_0(z)=\frac{4\pi G\rho_1}{\beta^2}(-e^{\beta z}+\beta z+1). \quad \ldots \quad (14)$$

Substituting trial functions for $w(z)$, $h(z)$, $\zeta(z)$ and $\xi(z)$ that satisfy the appropriate boundary conditions,

$$\left.\begin{aligned} w(z)&=A\sin az, & h(z)&=B\cos az\\ \zeta(z)&=E\cos az, & \xi(z)&=F\sin az\end{aligned}\right\}, \quad \ldots \quad (15)$$

in eq. (12) (with $i=j$), evaluating all the integrals contained therein, we obtain the dispersion relation :

$$\sigma^6\left\{\left(\frac{e^{m\pi a}-1}{m\pi a}\right)\left(\frac{1}{1+1/4a^2}\right)\left(1+x^2+\tfrac{1}{2}a^2\right)\right\}$$

$$+\sigma^4\left\{2L^2\left(\frac{e^{m\pi a}-1}{m\pi a}\right)\left(\frac{1+x^2+\frac{1}{2}a^2}{1+1/4a^2}\right)(1+x^2)\right.$$

$$+S\left[\left(\frac{e^{m\pi a}-1}{m\pi a}\right)\left(\frac{x^2}{1+1/4a^2}\right)-\left(\frac{e^{2m\pi a}-1}{2m\pi a}\right)\left(\frac{a^2+x^2}{1+a^2}+\frac{4a^3x}{(a^2+x^2+1-2ax)^2}\right)\right]$$

$$+U\left(\frac{e^{m\pi a}-1}{m\pi a}\right)\left(\frac{1+\frac{1}{2}a^2}{1+1/4a^2}\right)+3(x^2+1)\Big\}$$

$$+\sigma^2\Big\{L^4\left(\frac{e^{m\pi a}-1}{m\pi a}\right)(1+x^2)^2\left(\frac{1+x^2+\frac{1}{2}a^2}{1+1/4a^2}\right)+3L^2(1+x^2)^2$$

$$+2L^2S\Big[\left(\frac{e^{m\pi a}-1}{m\pi a}\right)\frac{x^2(1+x^2)}{(1+1/4a^2)}-\left(\frac{e^{2m\pi a}-1}{2m\pi a}\right)(1+x^2)$$

$$\times\left(\frac{a^2+x^2}{1+a^2}+\frac{4a^3x}{(1+a^2+x^2-2ax)^2}\right)\Big]$$

$$+2S\Big[x^2-\left(\frac{e^{m\pi a}-1}{m\pi a}\right)\left(\frac{a^2+x^2}{1+1/4a^2}+\frac{2a^3x}{(a^2+x^2-1-2ax)(1+x^2)}\right)\Big]$$

$$+3\left(\frac{e^{-m\pi a}-1}{-m\pi a}\right)\left(\frac{x^2+\frac{1}{2}a^2+1}{1+1/4a^2}\right)+2L^2U(1+x^2)\left(\frac{e^{m\pi a}-1}{m\pi a}\right)\left(\frac{1+\frac{1}{2}a^2}{1+1/4a^2}\right)$$

$$+2U^{1/2}L(1+x^2)+U\Big\}$$

$$+\Big\{L^4S\Big[\left(\frac{e^{m\pi a}-1}{m\pi a}\right)x^2(1+x^2)^2\frac{1}{1+1/4a^2}-\left(\frac{e^{2m\pi a}-1}{2m\pi a}\right)(1+x^2)^2$$

$$\times\left(\frac{a^2+x^2}{1+a^2}+\frac{4a^3x}{(1+a^2+x^2-2ax)^2}\right)\Big]+2L^2S\Big[x^2(1+x^2)-(1+x^2)\left(\frac{e^{m\pi a}-1}{m\pi a}\right)$$

$$\times\left(\frac{a^2+x^2}{1+1/4a^2}+\frac{2a^3x}{(1+a^2+x^2-2ax)(1+x^2)}\right)\Big]$$

$$+S\Big[\left(\frac{e^{-m\pi a}-1}{-m\pi a}\right)x^2\frac{1}{1+1/4a^2}-(x^2+a^2)\Big]$$

$$+L^4U(1+x^2)^2\left(\frac{e^{m\pi a}-1}{m\pi a}\right)\left(\frac{1+\frac{1}{2}a^2}{1+1/4a^2}\right)+2L^3U^{\frac{1}{2}}(1+x^2)^2+L^2U(1+x^2)$$

$$+L^2(1+x^2)^2\left(\frac{e^{-m\pi a}-1}{-m\pi a}\right)\left(\frac{1+\frac{1}{2}a^2}{1+1/4a^2}\right)+2LU^{\frac{1}{2}}\left(\frac{e^{-m\pi a}-1}{-m\pi a}\right)\left[\frac{1+x^2+\frac{1}{2}a^2x^2}{1+1/4a^2}\right]$$

$$+\left(\frac{e^{-2m\pi a}-1}{-2m\pi a}\right)\left(\frac{1+x^2+2a^2x^2}{1+a^2}\right)\Big\}=0, \qquad \ldots \quad (16)$$

where

$$\left.\begin{aligned}&\sigma=\frac{n}{\alpha V_1},\quad x=\frac{k}{\alpha},\quad a=\frac{\beta}{\alpha},\quad V_1^2=\frac{H_0^2}{4\pi\rho_1},\\ &S=\frac{4\pi G\rho_1}{\alpha^2V_1^2}\ \text{and}\ L=\frac{cH_0\alpha}{4\pi NeV_1}.\end{aligned}\right\}\qquad \ldots \quad (17)$$

To locate the roots of σ from eq. (16) against x, we have performed numerical calculations, for different values of the parameters L, S, m and a. These calculations are presented in figure 1, where we have plotted the growth rate σ against wave number x for different values of L, U and S, taking $m = 1{\cdot}0$ and $a = 0{\cdot}1$.

It is clearly seen from figure 1 that the growth rate of the unstable mode increases as either the Hall currents or rotation or both increase, exhibiting thereby the destabilizing character of the Hall currents as well as that of rotation. It is also seen from the figure that the effects of self-gravitations is stabilizing since growth rate(σ) decreases on increasing S.

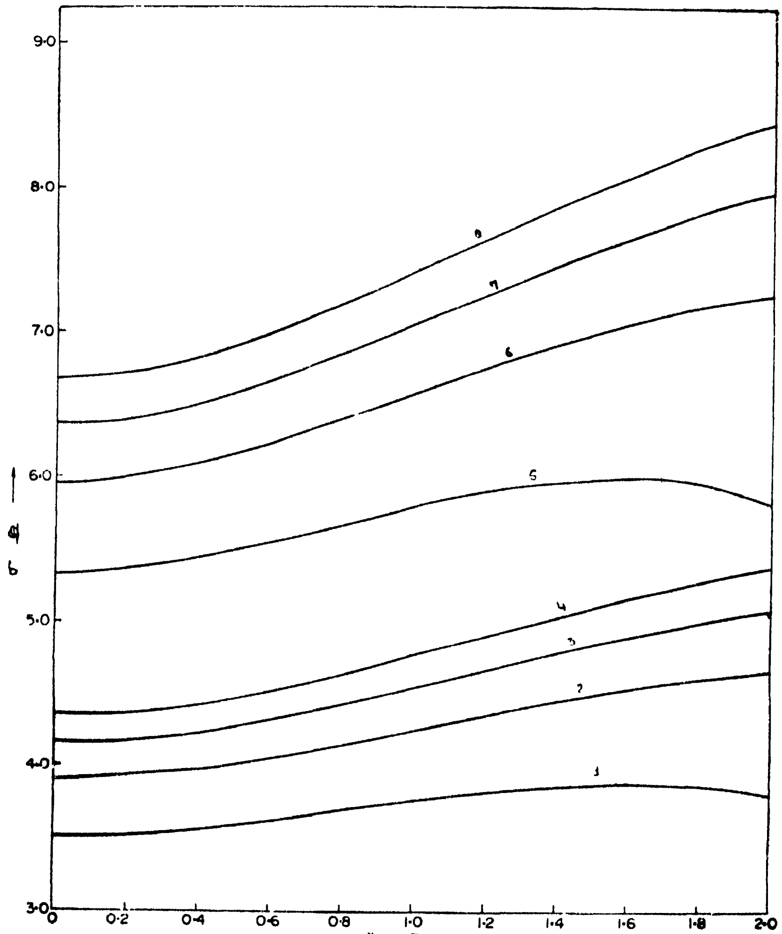

Fig. 1. Plot of growth rate σ against wave number x
Curves 1–4 are for $L = S = 5{\cdot}0$ and $U = 5{\cdot}0$, $10{\cdot}0$, $15{\cdot}0$, $20{\cdot}0$.
Curves 5–8 are for $L = 10{\cdot}0$, $S = 5{\cdot}0$ and $U = 5{\cdot}0$, $10{\cdot}0$, $15{\cdot}0$, $20{\cdot}0$.
In all the curves $m = 1{\cdot}0$, $a = 0{\cdot}1$,

We may thus conclude that rotation has a destabilizing influence on the hydromagnetic instability of an incompressible self-gravitating, Hall plasma of variable density.

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